

Exercise 5B

$$1 \text{ a } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix} = -6\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-6)^2 + (-6)^2 + (-3)^2} = \sqrt{81} = 9$$

$$\text{Area of triangle } OAB = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{9}{2} = 4.5$$

$$b \text{ } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -5 \\ 2 & 1 & -2 \end{vmatrix} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-3)^2 + (-4)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\text{Area of triangle } OAB = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{5\sqrt{2}}{2}$$

$$c \text{ } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 2 & 6 & -9 \end{vmatrix} = -27\mathbf{i} + 18\mathbf{j} + 6\mathbf{k} = 3(-9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$$

$$|\mathbf{a} \times \mathbf{b}| = 3\sqrt{(-9)^2 + 6^2 + 2^2} = 3\sqrt{121} = 33$$

$$\text{Area of triangle } OAB = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{33}{2} = 16.5$$

2 Find \overrightarrow{AB} and \overrightarrow{AC} and then use the formula: area of triangle $ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

$$a \text{ } \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 3 & -2 & 2 \end{vmatrix} = 8\mathbf{i} + 0\mathbf{j} - 12\mathbf{k}$$

$$\text{Area of triangle } ABC = \frac{1}{2} |8\mathbf{i} - 12\mathbf{k}| = \frac{1}{2} \sqrt{8^2 + (-12)^2} = \frac{1}{2} \sqrt{208} = \sqrt{52} = 2\sqrt{13}$$

$$b \text{ } \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -12 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 2 & -1 & -12 \end{vmatrix} = 12\mathbf{i} + 12\mathbf{j} + \mathbf{k}$$

$$\text{Area of triangle } ABC = \frac{1}{2} |12\mathbf{i} + 12\mathbf{j} + \mathbf{k}| = \frac{1}{2} \sqrt{12^2 + 12^2 + 1^2} = \frac{1}{2} \sqrt{289} = \frac{17}{2} = 8.5$$

$$3 \quad \overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -2 \\ 2 & -1 & -1 \end{vmatrix} = 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

$$\text{Area of triangle } ABC = \frac{1}{2} |-3\mathbf{j} + 3\mathbf{k}| = \frac{1}{2} \sqrt{(-3)^2 + 3^2} = \frac{1}{2} \sqrt{18} = \frac{3\sqrt{2}}{2}$$

$$4 \quad \overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$$

$$\text{Area of triangle } ABC = \frac{1}{2} |-5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}| = \frac{1}{2} \sqrt{(-5)^2 + 5^2 + 5^2} = \frac{1}{2} \sqrt{75} = \frac{5\sqrt{3}}{2}$$

5 Find \overrightarrow{AB} and \overrightarrow{AD} and then use the formula: area of parallelogram $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} - (\mathbf{i} + \mathbf{j} + \mathbf{k}) = -4\mathbf{i} + 3\mathbf{j}$$

$$\overrightarrow{AD} = 2\mathbf{i} - \mathbf{j} - (\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & 0 \\ 1 & -2 & -1 \end{vmatrix} = -3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$$

$$\text{Area of parallelogram } ABCD = |-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}| = \sqrt{(-3)^2 + (-4)^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$$

$$6 \quad \overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} \quad \overrightarrow{AD} = \begin{pmatrix} 1 \\ 6 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -4 \\ 1 & 1 & 3 \end{vmatrix} = -8\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$$

$$\text{Area of parallelogram } ABCD = |-8\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}| = \sqrt{(-8)^2 + (-10)^2 + 6^2} = \sqrt{200} = 2\sqrt{50} = 10\sqrt{2}$$

$$7 \quad \overrightarrow{AB} = \mathbf{i} + 4\mathbf{j} + \mathbf{k} - \mathbf{j} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\overrightarrow{AD} = 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k} - \mathbf{j} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & 5 & 3 \end{vmatrix} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$$

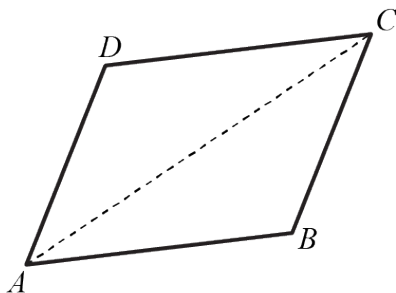
$$\text{Area of parallelogram } ABCD = |4\mathbf{i} - \mathbf{j} - \mathbf{k}| = \sqrt{4^2 + (-1)^2 + (-1)^2} = \sqrt{18} = 3\sqrt{2}$$

$$8 \quad \mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & a & 2a \\ 2a & a & 3a \end{vmatrix} = a^2\mathbf{i} + a^2\mathbf{j} - a^2\mathbf{k}$$

$$|\mathbf{p} \times \mathbf{q}| = \sqrt{(a^2)^2 + (a^2)^2 + (-a^2)^2} = \sqrt{3a^4} = \sqrt{3}a^2$$

$$\text{Area of triangle } OPQ = \frac{1}{2}|\mathbf{p} \times \mathbf{q}| = \frac{\sqrt{3}}{2}a^2$$

- 9 a The area of the parallelogram is twice the area of triangle ABC



$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} \quad \text{and} \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$$

$$\text{Area of triangle } ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$$

$$\Rightarrow \text{Area of parallelogram } ABCD = 2 \times \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})| = |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$$

$$b \quad (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})$$

$$\Rightarrow (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) - (\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) = \mathbf{0}$$

$$\Rightarrow (\mathbf{b} - \mathbf{a}) \times [(\mathbf{c} - \mathbf{a}) - (\mathbf{d} - \mathbf{a})] = \mathbf{0}$$

$$\Rightarrow (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{d}) = \mathbf{0}$$

This means that $\overrightarrow{AB} \times \overrightarrow{DC} = \mathbf{0}$, i.e. \overrightarrow{AB} is parallel to \overrightarrow{DC} .

$$10 \text{ a } \overrightarrow{AC} = (3\mathbf{i} + 3\mathbf{j}) - (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\overrightarrow{BC} = (3\mathbf{i} + 3\mathbf{j}) - (6\mathbf{i} - 2\mathbf{k}) = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{AC} \times \overrightarrow{BC} = (\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \times (-3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$

$$= -3(\mathbf{i} \times \mathbf{i}) + 3(\mathbf{i} \times \mathbf{j}) + 2(\mathbf{i} \times \mathbf{k}) - 12(\mathbf{j} \times \mathbf{i}) + 12(\mathbf{j} \times \mathbf{j}) + 8(\mathbf{j} \times \mathbf{k}) - 3(\mathbf{k} \times \mathbf{i}) + 3(\mathbf{k} \times \mathbf{j}) + 2(\mathbf{k} \times \mathbf{k})$$

$$= 3\mathbf{k} - 2\mathbf{j} + 12\mathbf{k} + 8\mathbf{i} - 3\mathbf{j} - 3\mathbf{i}$$

$$= 5\mathbf{i} - 5\mathbf{j} + 15\mathbf{k}$$

$$10 \text{ b } \text{Area of triangle } ABC = \frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{BC}| = \frac{1}{2} |5\mathbf{i} - 5\mathbf{j} + 15\mathbf{k}|$$

$$= \frac{1}{2} \sqrt{5^2 + (-5)^2 + 15^2} = \frac{\sqrt{275}}{2} = \frac{5\sqrt{11}}{2}$$

$$11 \text{ a } \overrightarrow{AB} = (-2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) - (-3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) = \mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$$

$$\overrightarrow{AC} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (-3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) = 4\mathbf{i} + 3\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}) \times (4\mathbf{i} + 3\mathbf{k})$$

$$= 4(\mathbf{i} \times \mathbf{i}) + 3(\mathbf{i} \times \mathbf{k}) - 20(\mathbf{j} \times \mathbf{i}) - 15(\mathbf{j} \times \mathbf{k}) + 20(\mathbf{k} \times \mathbf{i}) + 15(\mathbf{k} \times \mathbf{k})$$

$$= -3\mathbf{j} + 20\mathbf{k} - 15\mathbf{i} + 20\mathbf{j}$$

$$= -15\mathbf{i} + 17\mathbf{j} + 20\mathbf{k}$$

$$11 \text{ b } \text{Area of triangle } ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} |-15\mathbf{i} + 17\mathbf{j} + 20\mathbf{k}|$$

$$= \frac{1}{2} \sqrt{(-15)^2 + 17^2 + 20^2} = \frac{1}{2} \sqrt{225 + 289 + 400}$$

$$= \frac{\sqrt{914}}{2} = \frac{30.232}{2} = 15.12 \text{ m}^2 \text{ (2 d.p.)}$$

- c The area of fabric needed will be larger as there will need to be excess fabric to attach to the masts and some slack in the sail to fill with air.

$$12 \text{ a } C = D + \overrightarrow{AB}$$

$$= -2\mathbf{i} + 3\mathbf{k} + [(3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j})]$$

$$= -2\mathbf{i} + 3\mathbf{k} + 4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$$

So the coordinates of point C are $(2, -5, 1)$

12 b The area of the parallelogram $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

$$\overrightarrow{AB} = (3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$$

$$\overrightarrow{AD} = (-2\mathbf{i} + 3\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = (4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}) \times (-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

$$= -4(\mathbf{i} \times \mathbf{i}) - 8(\mathbf{i} \times \mathbf{j}) + 12(\mathbf{i} \times \mathbf{k}) + 5(\mathbf{j} \times \mathbf{i}) + 10(\mathbf{j} \times \mathbf{j}) - 15(\mathbf{j} \times \mathbf{k})$$

$$+ 2(\mathbf{k} \times \mathbf{i}) + 4(\mathbf{k} \times \mathbf{j}) - 6(\mathbf{k} \times \mathbf{k})$$

$$= -8\mathbf{k} - 12\mathbf{j} - 5\mathbf{k} - 15\mathbf{i} + 2\mathbf{j} - 4\mathbf{i}$$

$$= -19\mathbf{i} - 10\mathbf{j} - 13\mathbf{k}$$

$$\text{Area of the parallelogram } ABCD = |-19\mathbf{i} - 10\mathbf{j} - 13\mathbf{k}| = \sqrt{361 + 100 + 169} = \sqrt{630} = 3\sqrt{70}$$

$$\text{The volume of one pendant} = 0.3 \times 3\sqrt{70} = 0.9\sqrt{70} \text{ cm}^3$$

$$\text{So the cost to make one pendant} = 595 \times 0.9\sqrt{70} = \text{£}4480 \text{ to the nearest pound.}$$

Challenge

$$\text{Area } ABFE = |\overrightarrow{AB} \times \overrightarrow{BF}| = |\mathbf{p} \times (\mathbf{q} + \mathbf{r})| \quad \text{As } \overrightarrow{BF} = \mathbf{q} + \mathbf{r}$$

$$\text{Area } ABCD = |\overrightarrow{AB} \times \overrightarrow{BC}| = |\mathbf{p} \times \mathbf{q}|$$

$$\text{Area } CDEF = |\overrightarrow{CD} \times \overrightarrow{CF}| = |\mathbf{p} \times \mathbf{r}| \quad \text{As } \overrightarrow{CD} = \mathbf{p} \text{ (because } ABCD \text{ is a parallelogram)}$$

$$\text{Area } ABFE = \text{Area } ABCD + \text{Area } CDEF + \text{Area } ADE - \text{Area } BCF$$

$$\text{Using } \overrightarrow{AD} = \mathbf{q} \text{ and } \overrightarrow{DE} = \mathbf{r} \text{ (because } ABCD \text{ and } CDEF \text{ are parallelograms)}$$

$$\Rightarrow |\mathbf{p} \times (\mathbf{q} + \mathbf{r})| = |\mathbf{p} \times \mathbf{q}| + |\mathbf{p} \times \mathbf{r}| + \frac{1}{2}|\mathbf{q} \times \mathbf{r}| - \frac{1}{2}|\mathbf{q} \times \mathbf{r}|$$

$$\Rightarrow |\mathbf{p} \times (\mathbf{q} + \mathbf{r})| = |\mathbf{p} \times \mathbf{q}| + |\mathbf{p} \times \mathbf{r}|$$